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New Coupled Liouville System: Prolongation Structure, Soliton Solution, and Complete Integrability

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We suggest a new coupled Liouville equation which is exactly solvable. We obtain the Lax pair through a prolongation analysis and also obtain the exact one-soliton-like solution by a direct procedure. We confirm our result through a Painlevé analysis of the similarity reduced systems.

1. INTRODUCTION

The Liouville equation plays a central role in theoretical studies on string theory (Trieste, 1986) and nonlinear equations in two dimensions. The intimate connection of this equation with the infinite-dimensional Lie algebra is one of the most important properties of the equation. Here we suggest a coupled version of the Liouville equation which is proved to be completely integrable in the sense of Painlevé (Weiss, 1985), can sustain soliton-like solutions, and possesses a Lax pair.

2. FORMULATION

2.1. Prolongation Theory

The equations we study are

$$\begin{aligned} q_{xt} &= \mu F_x q_t + \mu^* F_t q_x \\ F_{xt} &= (\mu^* q_x q_t^* + \mu q_t q_x^*)\, e^{-4fF} \end{aligned} \tag{1}$$

(μ, f) are complex constants.

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For the derivation of the Lax pair, we follow the method of Whalquist and Estabrook (1975). We define a number of independent variables

$$\left.\begin{array}{lll} q_x = S, & q_x^* = S^*, & q_t = R \\ F_x = Z, & F_t = V, & q_t^* = R^* \end{array}\right\} \tag{2}$$

We now recast (1) and (2) in the language of differential forms,

$$\begin{aligned}
\alpha_1 &= dq \wedge dt - S\, dx \wedge dt \\
\alpha_2 &= dq \wedge dx + R\, dx \wedge dt \\
\alpha_3 &= dq^* \wedge dt - S^*\, dx \wedge dt \\
\alpha_4 &= dq^* \wedge dt + R^*\, dx \wedge dt \\
\alpha_5 &= dF \wedge dt - Z\, dx \wedge dt \\
\alpha_6 &= dF \wedge dx + v\, dx \wedge dt \\
\alpha_7 &= ds \wedge dx + \mu ZR\, dx \wedge dt + \mu^* VS\, dx \wedge dz \\
\alpha_8 &= dv \wedge dt + (\mu^* SR^* + \mu RS^*)\, e^{-4fF}\, dx \wedge dt \\
\alpha_9 &= dR \wedge dt + dS \wedge dx \\
\alpha_{10} &= dR^* \wedge dt + dS^* \wedge dx \\
\alpha_{11} &= dz \wedge dx + dv \wedge dt \\
\alpha_{12} &= dS^* \wedge dx + (\mu^* ZR^* + \mu VS^*)\, dx \wedge dt
\end{aligned} \tag{3}$$

The closure property of this set is easy to demonstrate. We now proceed to search for a differential one-form

$$\omega^j = dy^j + H^j\, dx + G^j\, dt \tag{4}$$

where

$$H^j = H^j(R, S, R^*, S^*, Z, v, q, q^*, F, x, t)$$
$$G^j = G^j(R, S, R^*, S^*, Z, v, q, q^*, F, x, t)$$

Imposing the closure on the exterior derivative of ω^j in the form

$$\begin{aligned} d\omega^j &= dH^j \wedge dx + dG^j \wedge dt \\ &= \sum g_i \alpha_i + (a\, dx + b\, dt) \wedge \omega^j \end{aligned}$$

we get

$$H_R = H_{R^*} = H_V = G_S = G_{S^*} = G_Z = 0 \tag{5}$$

along with

$$
\begin{aligned}
[H, G] = {} & G_q S - H_q R + G_{q^*} S^* - H_{q^*} R^* + G_F Z \\
& - H_F V - (\mu Z R + \mu^* V S)(H_S - G_R) - (\mu^* S R^* \\
& + \mu R S^*)\, e^{-4fF}(G_V - H_Z) - (\mu^* Z R^* + \mu V S^*) \\
& \times (H_{S^*} - G_{R^*})
\end{aligned}
\tag{6}
$$

We now set

$$
\begin{aligned}
H &= S H_1 + S^* H_2 + Z H_3 + H_4 \\
G &= R G_1 + R^* G_2 + v G_3 + G_4
\end{aligned}
$$

which immediately implies

$$
\begin{aligned}
[H_1, G_2] &= -\mu^*\, e^{-4fF}(G_3 - H_3) \\
[H_1, G_3] &= -H_{1F} - \mu^*(H_1 - G_1) \\
[H_2, G_1] &= -\mu\, e^{-4fF}(G_3 - H_3) \\
[H_4, G_3] &= -H_{4F} \\
[H_3, G_4] &= G_{4F} \\
[H_2, G_2] &= 0 \\
[H_2, G_3] &= -H_{2F} - \mu(H_2) - G_2) \\
[H_3, G_1] &= G_{1F} - \mu(H_1 - G_1) \\
[H_3, G_2] &= G_{2F} - \mu^*(H_2 - G_2) \\
[H_3, G_3] &= -H_{3F} + G_{3F}
\end{aligned}
\tag{7}
$$

where the subscript F stands for partial derivative with respect to it and

$$
[A, B] = \sum \left(A_i \frac{\delta B}{\delta y_k} - B_i \frac{\delta A}{\delta y_k} \right)
$$

The y_k are the prolongation variables. To reduce equations (7) to a set of algebraic commutation rules, we make the following choices:

$$
\begin{aligned}
H_4 &= e^{-4fF} X_4 \\
H_1 &= e^{-4fF} X_1 \\
G_1 &= e^{-4fF} Y_1
\end{aligned}
\tag{8}
$$

along with

$$
H_4 = G_4, \qquad H_3 = G_3 = X_3, \qquad H_2 = X_2, \qquad G_2 = Y_2
$$

where each X_i and Y_i are thought of as functions of the prolongation variables only, i.e., they depend on the y_k. With the prescription (8) we obtain immediately, from (7),

$$
\begin{aligned}
&[X_4, Y_1]=0, && [X_1, X_3]=4fX_1-\mu^*(X_1-Y_1)\\
&[X_4, Y_2]=0, && [X_2, Y_1]=0\\
&[X_2, X_4]=0, && [X_2, Y_2]=0\\
&[X_1, Y_2]=0, && [X_2, X_3]=-\mu(X_2-Y_2)\\
&[X_1, Y_1]=0, && [X_3, Y_1]=4fY_1-\mu(X_1-Y_1)\\
&[X_1, X_4]=0, && [X_3, Y_2]=-\mu^*(X_2-Y_2)\\
&[X_4, X_3]=4fX_4
\end{aligned} \tag{9}
$$

The next problem in the prolongation analysis is the closure of this Lie algebra (Roy Chowdhury, 1988), for which many techniques have been suggested, but none of them with a full proof. In the present case we have utilized the Jacobi identities and have tried to identify the unknown commutators.

We also make the observation that we already have two sets of closed sets of Lie algebra generated by $\{X_1, X_3, Y_1\}$ and $\{X_3, Y_2, X_2\}$, given as

$$
\begin{aligned}
&[X_1, X_3]=4fX_1-\mu^*(X_1-Y_1)\\
&[X_3, Y_1]=4fY_1-\mu(X_1-Y_1)\\
&[X_1, Y_1]=0
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
&[X_3, Y_2]=-\mu^*(X_2-Y_2)\\
&[X_2, X_3]=-\mu(X_2-Y_2)\\
&[X_2, Y_2]=0
\end{aligned} \tag{11}
$$

It is possible to map (10) to (11) by the choice $X_1=\alpha X_2$, $Y_1=\beta Y_2$ with $\mu=(\beta/\alpha)\mu^*$ or $\beta/\alpha=\mu/\mu^*$. Also, it is possible to set $X_4=X_6=0$. So we may write

$$
\begin{aligned}
H&=ZX_3+S^*X_2+S\,e^{-4fF}X_1\\
G&=X_3+R^*Y_2+R\,e^{-4fF}Y_1
\end{aligned} \tag{12}
$$

The algebra depicted in (10) and (11) can be represented by 3×3 or by 2×2 matrices, depending on the values and relations between the parameters μ, μ^*, and f.

2.2. Scaling Invariance

It is now interesting to note that our equations are invariant under the transformation $x' = \lambda x$, $t' = \sigma t$. Because under the scaling transformation, $H \to H' = H/\lambda$ and $G \to G' = G/\sigma$, we have

$$\begin{aligned}\omega' &= dy + H'\,dx' + G'\,dt' \\ &= dy + (H/\lambda)\,dx + (G/\sigma)\sigma\,dt \\ &= dy + H\,dx + G\,dt \\ &= \omega \end{aligned} \tag{13}$$

So the Lax pair does remain unchanged under this kind of transformation.

2.3. Explicit Wavelike Solutions

To search for progressive solutions we set

$$z = x - \eta t$$

whence equations (1) become

$$q_z = e^{(\mu+\mu^*)F+B} \tag{14}$$

$$F_{zz} = -(\mu + \mu^*)\, e^{\sigma F + 2B} \tag{14a}$$

Equation (14) can be integrated immediately and we obtain

$$F = \frac{1}{\sigma} \log\left[-\frac{c\sigma}{2(\mu+\mu^*)}\left(\coth^2 \frac{z\sqrt{c}}{2} - 1\right) - \frac{2B}{\sigma}\right] \tag{15}$$

Solution (15) when used in (13) yields

$$q = e^{-B'} \frac{c\sigma}{2(\mu+\mu^*)} \int \left(\operatorname{cosec}^2\mathrm{h}\, \frac{z\sqrt{c}}{2}\right)^{(\mu+\mu^*)/\sigma} dz \tag{16}$$

The quadrature in (16) can be performed for some specific values of $\mu + \mu^*/\sigma$ and in particular for $(\mu + \mu^*)/\sigma = -1$, in which case we obtain

$$q = -e^{-B'} \frac{c\sigma}{4(\mu+\mu^*)}\left(\frac{1}{\sqrt{c}} \sinh z\sqrt{c} - z\right)$$

In the above expressions for these solutions we have used

$$B' = B\left(1 - \frac{2}{\sigma}\right)(\mu + \mu^*), \qquad \sigma = 2(\mu + \mu^*) - 4f$$

2.4. Complete Integrability and Painlevé Analysis

Any nonlinear system, if it possesses a Lax pair, is usually associated with an infinite number of conservation laws, and such systems are known to be completely integrable. In the present case we now apply the Painlevé criterion as advocated by Ablowitz *et al.* (1980) to test the complete integrability of our system. According to Ablowitz *et al.* (1980), a partial differential nonlinear system is to be termed completely integrable if and only if all its possible symmetry-reduced ordinary nonlinear equations pass the Painlevé test.

Our system does have translation symmetry which dictates the propagating wave, and since the reduced equation (14) is already of Liouville type, nothing remains to be tested.

A second invariance is the scale invariance as noted in Section 2.2. So we must have a solution of the form

$$F(x, t) = F'(\xi); \qquad q(x, t) = q'(\xi); \qquad \xi = x/t \tag{17}$$

Transforming the original system to this new variable, we get

$$q'_{\xi\xi} + \frac{q'_{\xi}}{\xi} = (\mu + \mu^*) q'_{\xi} F'_{\xi}$$

$$F'_{\xi\xi} + \frac{F'_{\xi}}{\xi} = -(\mu + \mu^*) q'_{\xi} q^*_{\xi}\, e^{-4fF'} \tag{19}$$

Integrating equation (18), we get

$$q'_{\xi} = \frac{1}{\xi}\, e^{(\mu+\mu^*)F'} \tag{20}$$

and if we set $e^{\sigma F'} = g$, then we arrive at

$$\xi^2 g g_{\xi\xi} - \xi^2 g_{\xi}^2 + \xi g g_{\xi} = -\sigma(\mu + \mu^*) g^3 \tag{21}$$

So equation (21) is the required nonlinear ordinary differential equation for which the Painlevé analysis is to be performed.

The basic philosophy of a Painlevé test is to assume an expansion of the solution g over the solution manifold. In our case we set

$$g \sim H_0(\xi - \xi_0) = H_0 \phi^{\alpha}$$

By comparing the most singular terms in equation (21), we get

$$\alpha = -2, \qquad H_0 = -\frac{2}{K}\xi_0^2 \tag{22}$$

with $K=\sigma(\mu+\mu^*)$. Now we set the expansion

$$y=\sum_{j=0}^{\infty} H_j\phi^{j-2}$$

But before comparing powers of $(\xi-\xi_0)$, we must write

$$\xi^2=(\phi+\xi_0)^2=\phi^2+2\phi\xi_0+\xi_0^2$$

Equating the same powers of ϕ, we get

$$\begin{aligned}
&[\xi_0^2H_0(n-2)(n-3)+6\xi_0^2H_0+4\xi_0^2H_0(n-2)+3KH_0^2]H_n\\
&+\left\{\xi_0^2\sum_{s=1}^{n-1} H_sH_{n-s}(s-2)(s-3)\right.\\
&-\xi_0^2\sum_{s=1}^{n-1} H_sH_{n-s}(s-2)(n-s-2)\\
&+K\sum_{s=0}^{n-1}\sum_{p=0}^{n-1} H_sH_pH_{n-s-p}+\sum_{s=0}^{n-2} H_sH_{n-s-2}(s-2)(s-3)\\
&+2\xi_0\sum_{s=0}^{n-1} H_sH_{n-s-1}(s-2(s-3)\sum_{s=0}^{n-2} H_sH_{n-s-2}(s-2)(n-s-4)\\
&-2\xi_0X\sum_{s=0}^{n-1} H_sH_{n-s-1}(s-2)(n-s-3)\\
&\left.+\sum_{s=0}^{n-2} H_sH_{n-s-2}(s-2)+\xi_0\sum_{s=0}^{n-1} H_sH_{n-s-1}(s-2)\right\}=0
\end{aligned}\tag{23}$$

Resonance positions occur through the zeros of $\Delta=0$, where

$$\begin{aligned}
\Delta&=\xi_0^2H_0(r-2)(r-3)+\xi_0^2H_0+4\xi_0^2H_0(r-2)+3KH_0^2\\
&=\xi_0^2H_0(r-2)(r+1)
\end{aligned}\tag{24}$$

so at $r=-1,2$. The resonance at $r=-1$ signifies that the expansion point ξ_0 is perfectly arbitrary. Since our equation is of second order, it cannot have more than two resonances. Now we check the arbitrariness of the coefficient of expansion at $r=2$.

First, for $n=1$ equation (23) leads to

$$H_1=-\frac{2}{K}\xi_0 \tag{25}$$

Note that $r=1$ is not a resonance position, so the coefficient at $r=1$ is fixed. Now for $n=2$ in (23) we get

$$0\cdot H_2-H_15\xi_0^2\left(H_1-\frac{H_0}{\xi_0}\right)=0$$

so that the coefficient cannot be determined and we get the condition $H_1 = H_0/\xi_0$, which is identically satisfied by the values noted in equations (22) and (25). So we can ascertain that the proposed nonlinear system is completely integrable in the true sense of the term.

3. DISCUSSION

We have presented a detailed analysis of a new nonlinear system, the coupled Liouville system. The Lax pair has been obtained by a prolongation approach, the soliton-like solution can be deduced directly, and also the complete integrability can be ascertained via a Painlevé analysis. Some comment may be in order about the closer of the Lie algebra. There is no unique way for obtaining such a closer and various methods may lead to various kinds of Lax equations for the same nonlinear equation.

REFERENCES

Ablowitz, M. J., Ramani, R., and Segur, H. (1980). *Journal of Mathematical Physics*, **21**, 715.

Roy Chowdhury, A. (1988). Lecture Given at the Workshop at Tiruchirapalli, Tamilnadu (India), in *Nonlinear Dynamics Proceedings*, Springer-Verlag, New York.

Trieste (1986). *Supersymmetry, Supergravity, Superstrings* (Proceedings of the Trieste Spring School, 7-15 April 1986), World Scientific, Singapore.

Weiss, J. (1985). *Journal of Mathematical Physics*, **26**, 258.

Whalquist, H., and Estabrook, F. B. (1975). *Journal of Mathematical Physics*, **16**, 1.